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Know Mathematics "In Depth"
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UNDERSTANDING HIGH SCHOOL BODIES OF KNOWLEDGE

What Does It Mean to Know Mathematics "In Depth"?

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The Next Generation Sunshine State Standards (NGSSS) for mathematics were designed to address the perceived problem that students are expected to learn too many topics at every grade level (Schoen & Clark, 2007). The phrase "a mile wide and an inch deep" has been used to describe U.S. mathematics standards. The 1996 Sunshine State Standards, with an average of 83 grade level expectations per grade between Kindergarten and eighth grade, were often used to illustrate this "mile wide" curriculum. In contrast, language of the NGSSS includes expectations for students to create models of mathematical objects and concepts, explain and justify various algorithms and procedures, and develop quick recall of some mathematics facts, such as the basic multiplication facts and their related division facts (FLDOE, 2007).

Although the NGSSS are meant to provide more opportunity for students to learn mathematics with more depth, classroom teachers are charged with selecting or designing appropriate tasks to teach mathematics in depth. Dixon (2008) raises an important question, "How will curriculum implementation be revised so that teachers find the time to incorporate more strategies and depth for mathematical topics?" (p. 8). We believe that the concept of cognitive complexity, or Depth of Knowledge (Webb, 1997), provides a structure through which to consider *depth* in teaching and learning mathematics. In fact, each benchmark in the NGSSS has been assigned a target for the cognitive complexity of student tasks (FLDOE, 2007). The FLDOE uses three of the four levels in Webb's Depth of Knowledge framework and the rating of each benchmark provides a ceiling for the state assessment related to that benchmark. Thus, students need to engage in tasks related to the benchmarks and Big Ideas from the NGSSS at sufficiently high levels of cognitive complexity to be prepared to perform well on the state assessment. In this article, we illustrate tasks on the same topic that invoke a range of cognitive complexity and we propose strategies for generating tasks with different levels of cognitive complexity. These strategies may be utilized to convert tasks at a low level of cognitive complexity into ones at a higher depth of knowledge.

Levels of Complexity within Quadratic Equations

Low level tasks. To introduce levels of cognitive complexity (or Depth of Knowledge) for high school mathematics, we consider the topic of quadratic functions. Tasks with low cognitive complexity are procedural or routine tasks that often require recall of information. In a low complexity task, students are encouraged to follow an algorithm or apply a formula that has already been presented to them. Low complexity tasks may simply require students to recall definitions or procedures or retrieve information from a graph or table. For instance, a task that requires a student to "State the quadratic formula" is an example of a task that predominantly requires recall of information. Other examples of these tasks for quadratic functions follow:

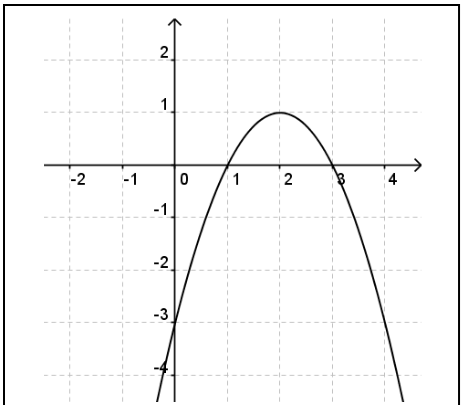

Figure 1

1. Given the graph of $y = -(x - 2)^2 + 1$ (see Figure 1), determine the axis of symmetry, vertex, and x-intercepts.

2. Use the discriminant to determine the nature and number of solutions of the quadratic equation $x^2 - 5x + 4 = 0$.

Moderate level tasks. Tasks with a moderate level of cognitive complexity involve multiple solution methods or translation between representations. The following tasks exemplify a moderate level of cognitive complexity:

3. Describe how you can draw the graph of $y = -(x - 2)^2 + 1$ from the graph of $y = x^2$ and determine the axis of symmetry, vertex, and intercepts of the new graph.

4. Determine the nature and number of solutions of the quadratic equation $x^2 - 5x + 4 = 0$ without using the discriminant.

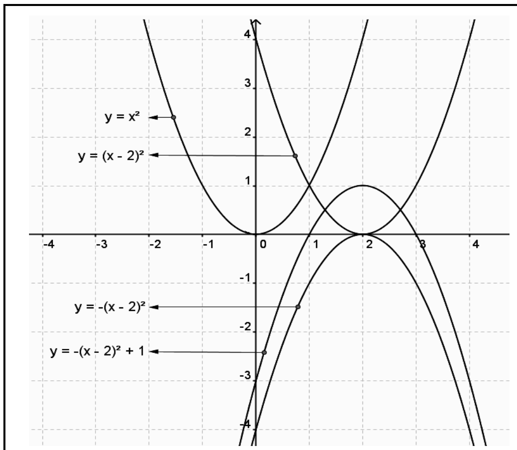

Figure 2

The cognitive complexity of items 3 and 4 is higher than in items 1 and 2 because possible solution methods are not explicit in tasks 3 and 4. Students must make their own decisions regarding how to approach items 3 and 4.

Consider task three. Drawing the graph of $y = -(x - 2)^2 + 1$ with technology is a routine task and requires a low level of cognitive complexity. When students are provided with rules to graph equations given in a specific format, the task may also be completed at a low cognitive complexity level without using technology. However, requiring a student to think about the relationship between the graph of $y = -(x - 2)^2 + 1$ and the graph of $y = x^2$ raises the level of the task to a moderate cognitive complexity. To complete the task, the student may think that the graph of $y = x^2$ is shifted up one unit in the positive y-direction, and then that the negative sign turns a “happy parabola” to a “sad parabola.” Interpreting the effect of $(x - 2)$ may be slightly more difficult to intuit. A student may rely on a memorized rule, reason that the maximum value of the function occurs when $(x - 2) = 0$, or reason that the value that is squared is two more than for the case of $y = x^2$. In any case, the graph of $y = -x^2 + 1$ is shifted to the right two units to complete the transformation to $y = -(x - 2)^2 + 1$. Figure 2 arrives at the same result through a different order of steps.

Consider task four. Determining the nature of the solution of a quadratic equation $x^2 - 5x + 4 = 0$ by using the discriminant requires a low level of cognitive complexity because students only need to apply a specific procedure. However, when given the condition without using the discriminant, students must synthesize alternative strategies to solve the task. For instance, the solutions, if any, of the quadratic equation $x^2 - 5x + 4 = 0$ will be the x-intercepts for $y = x^2 - 5x + 4$. To draw the graph, students may determine the coordinates of the vertex $V(h, k)$. One approach is to complete the square as follows: $x^2 - 5x + 4 = x^2 - 5x + \frac{25}{4} - \frac{25}{4} + 4 = (x - \frac{5}{2})^2 - \frac{9}{4}$. At this point, the student could use the same reasoning as the solutions shared for task one. Alternatively, the student may find the coordinates of the vertex using two familiar formulas:

$$h = \frac{-b}{2a} = \frac{-(-5)}{2} = \frac{5}{2} \quad \text{and} \quad k = \frac{4ac - b^2}{4a} = \frac{4(1)(4) - (-5)^2}{4(1)} = \frac{16 - 25}{4} = \frac{-9}{4}.$$

Because the vertex $V(\frac{5}{2}, \frac{-9}{4})$ is in the fourth quadrant and the parabola opens upwards (i.e., the coefficient of x^2 is positive), the parabola crosses the x-axis at two points ($x = 1$ and $x = 4$). Thus, the quadratic equation has exactly two real solutions (Figure 3).

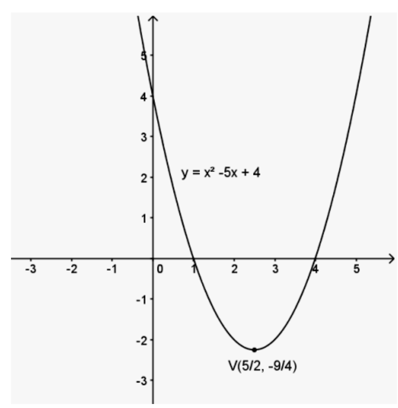

Figure 3

High level tasks. Tasks with high cognitive complexity necessarily involve non-routine tasks and can usually be approached from multiple solution methods. Non-routine means the solution methods are not explicitly suggested in these tasks or in recent tasks that have been posed or completed. Asking students to make connections between concepts or representations, synthesize ideas, or explore a variety of strategies to solve high cognitive level tasks creates rich opportunities for students to engage in doing mathematics and in learning how to think (not just what to think). For instance, solving real-world problems with quadratic equations can be classified as moderate or high cognitive complexity depending on the task. Consider the following:

5. The area, A, of a rectangular field is given by $A(x) = 1000 - 2x^2$. What is the maximum area of the field?

6. A farmer has 2000 feet of fencing available to enclose a rectangular field and divide it into three rectangular sub-plots. One configuration of this rectangular field and sub-plots is pictured in Figure 4. What are the dimensions (x and y) of the largest rectangular field that the farmer can create?

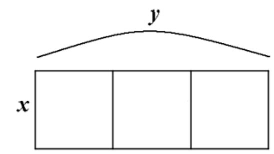

Figure 4

In task five, students find the maximum value of the quadratic function A(x). However, the task does not identify the function as quadratic. A student may reason that the $-2x^2$ term is always negative or zero. Thus, the maximum area occurs when $-2x^2 = 0$. Solving the problem with this line of reasoning may not require any consideration of quadratics or vertices, but a more fundamental recognition that the square of a real number is always positive or zero. This task poses a question related to quadratic functions in a non-routine manner. The task may be solved in different ways using different mathematical concepts. Thus, the task is a moderate or high complexity task.

To solve task six, students need to express the length of fencing used and the area in terms of x and y (i.e., *Total Length of Fencing* = 2000 = $4x + 2y$ and *Area of the Field* = xy). After solving $4x + 2y = 2000$ for $2y$, the expression $1000 - 2x$ may be substituted for y in the equation *Area* = $xy = x(1000 - 2x) = 1000x - 2x^2$. The student must now recognize that this function is quadratic, so the x-value of the vertex provides the length of one side of the fence; evaluating the function for that value provides the area. Using these two values for *Area* and x, the student may elect to solve the equation *Area* = xy for y. There are other, less efficient ways to solve this problem without using calculus, such as substituting the x-value of the vertex into $4x + 2y = 2000$ and solving for y. Either solution is valid; the former may indicate a stronger understanding of function, while the latter may indicate a stronger understanding of solving equations. Still, the student must synthesize all of the information extracted from the problem to determine the answer to the initial question: What are the dimensions of the rectangular field?

Tasks with moderate or high complexity are not difficult to develop. Without spending a significant amount of time, a teacher can convert routine tasks with low cognitive complexity into moderate or high cognitive complexity tasks. Modifications are often as simple as asking students to answer the questions “Why does that work?” or “Why does that make sense?”

Conclusion

When we classify some of the tasks as procedural with low cognitive complexity, we do not claim these tasks are not beneficial or should be given decreased attention. On the contrary, we think that tasks with low cognitive complexity serve a crucial role in building students' mastery of basic algorithms and mathematical facts. But teaching how to solve only procedural tasks (i.e., using low cognitive complexity) is not a sufficient goal or desired end point for instruction, because a singular focus on practicing prescribed procedures defeats the purpose of teaching students how to think and reason. For many students, understanding and facile thinking does not occur through the mastery of procedures and algorithms. To teach and learn mathematics deeply, it is important that all students approach a variety of problems with high, low, and moderate levels of cognitive complexity.

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