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UNDERSTANDING THE HIGH SCHOOL BODIES OF KNOWLEDGE: CALCULUS

Mathematical Continuity: Identifying, Exposing, and Closing the Gaps of Understanding

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Students' understandings of functions and graphs have been at the center of numerous research studies with the concept of continuity an essential component of this understanding. We believe that students should make connections between algebraic and graphic definitions of continuity as they explore graphs of functions. The National Council of Teachers of Mathematics (NCTM, 2000) advocates the use of multiple representations and connections at all levels of mathematics instruction. For Lesh, Post and Behr (1987), *understanding* an idea is the ability to recognize the idea embedded in different representations, to manipulate the idea within given representations, and to translate the idea from one representation to another. In this article, our goal is to provide an approach to teaching the mathematical concept of continuity that will encourage students to explore this concept using different representations to enhance their understanding.

Examples to Introduce Continuity

Students construct rich understanding of a concept when they explore and synthesize the relationships among multiple representations (NCTM, 2000). If the meaning of the word *continuous* in daily language is combined with a graphical approach to the concept of continuity, a continuous function on an interval can be informally defined as a function whose graph has no breaks, jumps, or holes in that interval. Following Hughes-Hallett's (2002, p. 45) definition, "A continuous function has a graph which can be drawn without lifting the pencil from the paper." Using this definition, the graph in Figure 1 is continuous on the interval $x \in (-2,2)$. Algebraically, a continuous function is defined using limits: "A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$" (Stewart, 2003, p. 124). We can use this definition to show the function is continuous on the interval $x \in (-2,2)$. Because the limit as x approaches zero, $\lim_{x \to 0} f(x)$, exists and is equal to the value of the function at that point, $f(0)$, then f is continuous at $x = 0$. The same reasoning applies for any other point on the function corresponding to a point in the domain.

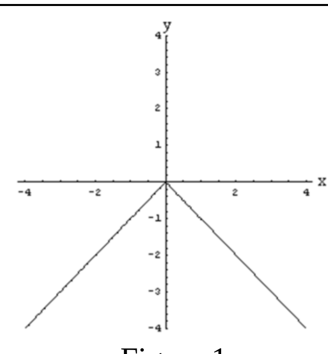


Figure 1

Consider the discontinuous function in Figure 2 from both algebraic and graphic points of view. Using the graphic definition of continuity, the function in Figure 2 is not continuous at $x = 0$; although it can be traced from either direction toward $f(0)$, the pencil could not cross the hole without being lifted from the paper. Algebraically, although the limit at $x = 0$ exists, the function is not continuous at $x = 0$ because $f(0)$ is not defined.

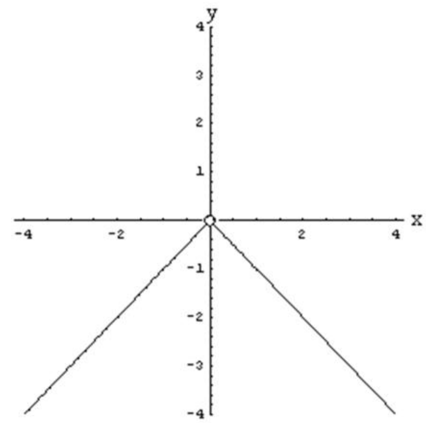


Figure 2

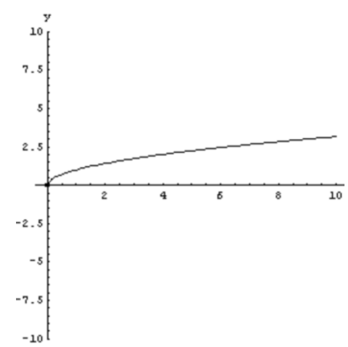
Even though these examples may serve to introduce the concept of continuity, we found them to be contrived or over-simplified. They do not challenge a student's basic understanding of mathematical continuity.

The mathematical definition of continuity has little bearing on students' interpretation of whether these functions are continuous. We asked high school Calculus, Precalculus, and Algebra students to identify the graphs in Figures 1 and 2 as either continuous or not continuous. All students from all three cohorts labeled them correctly. In our experience, high school students may determine that the graph in Figure 1 represents a function that is continuous but the function whose graph is in Figure 2 is not continuous without considering a formal mathematical definition of continuity. It was not necessary for students to consult a mathematical definition of continuity; they could use the everyday meaning of the term *continuous* to analyze the functions.

Challenging Functions for Calculus Students

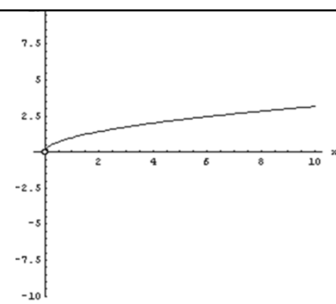
At this point, it is important to engage students in a deeper, more critical discussion of the mathematical concept of continuity. As educators, we are obliged to provide challenging tasks that provoke students to explore the formal definition and struggle with their understanding. The learning opportunity arises when students experience a "felt gap" in their understanding and see the value in using different approaches (e.g., graphic or algebraic). For example, after introducing the concept of continuity to students with the examples and definitions from Figures 1 and 2, we formed heterogeneous pairs of students, with one visual and one analytic learner in each pair, to explore more challenging functions. Students were presented with the graphs in Figures 3 and 4 and the following instructions:

Are these functions continuous on their domains? Provide reasons for your answers.



$f:[0,\infty)\rightarrow R, f(x)=\sqrt{x}$

Figure 3



$f:(0,\infty)\rightarrow R, f(x)=\sqrt{x}$

Figure 4

The examples in Figures 3 and 4 have less obvious results regarding continuity than the examples in Figures 1 and 2. Although ostensibly similar, each function may be understood using different definitions of continuity – graphic or algebraic – that may lead students to different and sometimes divergent interpretations for the same function. For the functions and graphs in Figures 3 and 4, a conclusion requires more thinking and justification than in the previous examples. In fact, a conclusive argument requires a more extensive definition than either the Hughes-Hallet or Stewart definitions. Consider the definition of Ostebee and Zorn (2002, p.106): "If the point happens to be the left or right end point of the domain of *f*, then a right-hand limit or a left-hand limit replaces the two sided limit in the definition." Therefore, the student must conclude that the function in Figure 3, $f(x)=\sqrt{x}$, is continuous at zero although the left-hand limit does not exist at $x = 0$. The function in Figure 4 is continuous on its domain, $(0, \infty)$, because for any point in the domain of the function, the limit exists and equals the value of the function at that point.

Students with different learning styles utilize distinct strategies. As they work together on the tasks and understand each others' reasoning, it is possible they will enhance their conceptual understanding and resolve possible cognitive dissonance (Burris, Heubert, & Levin, 2006). Our students exchanged ideas and shared their results and solution strategies first with their partners and later during a whole class discussion. The small group discussion helped them become more confident in their reasoning before sharing with the whole class. During discussion, our role as teachers was simply to orchestrate the discussion without betraying what we felt the answer should be. We did not want students to stop thinking about their own understanding.

During our discussions with high school Calculus students, we observed that students who used only the algebraic or the graphic definition of continuity experienced difficulties. For example, for students who preferred the graphic definition of continuity, the inclusive and exclusive end points indicated continuity and discontinuity *at* $x = 0$ respectively, "This one [in Figure 3] is continuous because it is inclusive at $x = 0$. There is an actual value there. This one [in Figure 4] is discontinuous because it is exclusive at $x = 0$." In contrast, students who preferred the algebraic definition of continuity encountered different difficulties. They thought that both functions were discontinuous at $x = 0$ because the graphs are not defined for negative *x* values, "Both graphs are not continuous since $\lim_{x\to 0^-} f(x)$ does not exist for either of them."

For an even deeper challenging function to investigate, consider introducing the *monsters* (Lakoff & Nunez, 2000) in Figures 5 and 6 to provide two examples of rich functions to perturb student thinking and provoke discussion of the formal mathematical definition of continuity, thereby deepening student understanding. The monster in Figure 5 is not continuous; at every point *x*, there is a point infinitely close where the function jumps. It is not possible to draw an accurate graph, but the process of trying to draw it is important for students to support their thinking. The monster in Figure 6 (Tall & Vinner, 1981) might conflict with students' concept image or concept definition of continuous functions. Its graph in Figure 7 shows that this function is continuous despite the gap in the graph at $x=\sqrt{2}$ because the function is defined on the rational numbers. The process of examining these

$$f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

For what values of *x* is *f(x)* continuous?

Figure 5

$$f: Q \to Q, f(x)=\begin{cases}0 & \text{if } x<0 \text{ or } x^2<2\\ 1 & \text{if } x>0 \text{ and } x^2>2\end{cases}$$

Figure 6

functions both graphically and algebraically provides rich fodder for discussion and encourages students to examine both representations.

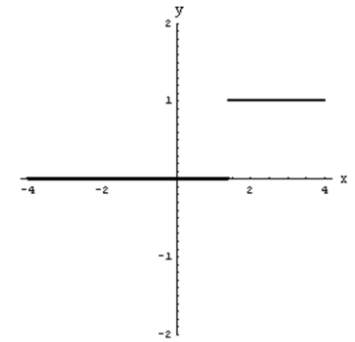


Figure 7

Conclusion

Our teaching experiments with students have been successful, and we recommend you pose these functions and graphs to your students to initiate discussion. As your students negotiate the continuity of these functions, the classroom discussion will likely be interesting and create an opportunity for a deeper, shared understanding of mathematical continuity.

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